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On the Asymptotic Behavior of Trigonometric Series. I

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Suppose that $a_n \geqslant 0$, and $f(x)$ and $g(x)$ are defined by

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx,$$

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx,$$

wherever the series converge. The asymptotic behavior of $f(x)$ and $g(x)$ (as $x \to +0$) for several orders of magnitude of the a_n's is already known in the literature (see [1–5]). In this paper, we shall establish the following two theorems:

THEOREM 1. *Suppose that* $\alpha > 2$, $\alpha \neq 2i$ $(i = 1, 2,...)$. *Then*[1]

$$A_n = \sum_{k=n}^{\infty} a_k \simeq \frac{1}{\alpha - 1} n^{1-\alpha} L(n) \qquad as\ n \to \infty \tag{1a}$$

if and only if

$$g(x) - \sum_{j=1}^{j=[\alpha/2]} \frac{1}{(2j-1)!} g^{(2j-1)}(0)\, x^{2j-1} \simeq \frac{\pi}{2\Gamma(\alpha)\sin(\alpha\pi/2)} x^{\alpha-1} L\left(\frac{1}{x}\right) \qquad as \quad x \to +0. \tag{2}$$

[1] $f_1(x) \simeq f_2(x)$, as $x \to a$, means that $f_1(x)/f_2(x) \to 1$ as $x \to a$.
$L(x)$ represents a measurable slowly varying function (see (4a) and (4b)).
$g^{(j)}(x)$ and $f^{(j)}(x)$ are jth derivatives of $g(x)$ and $f(x)$; in particular, $g^{(2j-1)}(0) = (-1)^{j-1} \sum_{k=1}^{\infty} k^{2j-1} a_k$ and $f^{(2j)}(0) = (-1)^j \sum_{k=1}^{\infty} k^{2j} a_k$.
A sequence $\{a_n\}$ is said to be quasi-monotone if $a_{n+1} \leqslant a_n\{1 + (\beta/n)\}$, or equivalently, $\{n^{-\beta} a_n\}$ decreases monotonically for certain $\beta > 0$ [6].
Any one assumption of (1a), (1b), (2) and afterwards (3) ensures the convergence of $\sum_{n=1}^{\infty} a_n$.

If, in addition, $\{a_n\}$ *is monotone (or quasi-monotone),* (1a) *can be replaced by*

$$a_n \simeq n^{-\alpha}L(n) \qquad as \quad n \to \infty. \tag{1b}$$

THEOREM 2. *Suppose that* $\alpha > 1$, $\alpha \neq 2i - 1$ $(i = 1, 2, \ldots)$. *Then*

$$A_n = \sum_{k=n}^{\infty} a_k \simeq \frac{1}{\alpha - 1} n^{1-\alpha} L(n) \qquad as \quad n \to \infty$$

if and only if

$$f(x) - \sum_{j=0}^{j=[\frac{\alpha-1}{2}]} \frac{1}{(2j)!} f^{(2j)}(0)\, x^{2j} \simeq \frac{\pi}{2\Gamma(\alpha)\cos(\alpha\pi/2)} x^{\alpha-1} L\left(\frac{1}{x}\right) \quad as \quad x \to +0. \tag{3}$$

If, in addition, $\{a_n\}$ *is monotone (or quasi-monotone),* (1a) *can be replaced by* (1b).

The necessity of Theorems 1 and 2 make an improvement on Adamović's theorem (see conditions (C_1), (C_2), and (C_1') in Theorem 1 of [1]). The sufficiency of Theorems 1 and 2 is an extension (for large α) and also a generalization (with $L(x)$) of Boas' theorems (see Theorems 1 and 2 of [3]).

A positive and measurable function is defined to be slowly varying if it satisfies

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1 \tag{4a}$$

for every fixed $\lambda > 0$. When the asymptotic relation (4a) holds for every $\lambda > 0$, then actually it holds uniformly for λ on any finite interval $0 < a \leqslant \lambda \leqslant b < \infty$. A slowly varying function also can be represented[2] by

$$L(x) = c(x) \exp \int_1^x \frac{\epsilon(t)}{t}\, dt, \tag{4b}$$

where $c(x)$ is a positive and measurable function such that $\lim_{x\to\infty} c(x) = c > 0$ and $\epsilon(x)$ is continuous and $\lim_{x\to\infty} \epsilon(x) = 0$ [4]. Other properties may be found in [4] and [7]. By way of explanation of the interest of the modification of Boas' theorems with $L(x)$, we observe that a slowly varying function may approach either a finite limit c $(c > 0)$, or infinity, or it may swing continuously back and forward between arbitrarily large and arbitrarily small positive values as $x \to \infty$. The last characteristic is illustrated by the following example:

[2] Without loss of generality, $L(x)$ may be considered only for $x \geqslant 1$.

Construct

$$L^*(x) = \exp \int_1^x \frac{\epsilon(t)}{t}\, dt$$
$$= \exp \int_1^{x_1} \frac{\epsilon_0(t)}{t}\, dt \cdot \exp \int_{x_1}^{x_2} \frac{\epsilon_1(t)}{t}\, dt \cdots \exp \int_{x_n}^{x} \frac{\epsilon_n(t)}{t}\, dt,$$

where $\epsilon_0(t) = 1$ and x_1 is so large that $L^*(x_1) = D_1 > 1$;

$$\epsilon_1(t) \begin{cases} \text{linear from } 1 \text{ to } -\frac{1}{2} \text{ for } x_1 \leqslant t \leqslant x_1 + 1 \\ = -\frac{1}{2} \text{ for } x_1 + 1 \leqslant t \leqslant x_2 \text{ where } x_2 \text{ is so large that } L^*(x_2) = D_2 < \frac{1}{2}; \end{cases}$$

$$\epsilon_2(t) \begin{cases} \text{linear from } -\frac{1}{2} \text{ to } +\frac{1}{2} \text{ for } x_2 \leqslant t \leqslant x_2 + 1 \\ = +\frac{1}{2} \text{ for } x_2 + 1 \leqslant t \leqslant x_3 \text{, where } x_3 \text{ is so large that } L^*(x_3) = D_3 > 2; \end{cases}$$

and so on. Thus we obtain a slowly varying function such that $L(x_{2K}) < 1/(2k)$ and $L(x_{2K+1}) > 2k$, where k is arbitrary.

Lemma 1. *Suppose that $n^{-\eta}a_n$ approaches zero monotonically for certain $\eta > 0$. Then for arbitrary $\alpha > 0$,*

$$\sum_{k=n}^{\infty} a_k \simeq n^{-\alpha}L(n) \qquad \text{as} \quad n \to \infty,$$

implies that

$$a_n \simeq \alpha n^{-\alpha-1}L(n) \qquad \text{as} \quad n \to \infty.$$

Proof of Lemma 1. Write

$$\sum_{k=n}^{\infty} a_k = n^{-\alpha}L_1(n) \simeq n^{-\alpha}L(n) \qquad \text{as} \quad n \to \infty.$$

Let $n \leqslant m = [n + \delta n]$ for $\delta > 0$. Then

$$\sum_{k=n}^{m} a_k = n^{-\alpha}L_1(n) \left\{1 - \left(\frac{m+1}{n}\right)^{-\alpha} \frac{L_1(m+1)}{L_1(n)}\right\}$$
$$= n^{-\alpha}L_1(n)\{1 - (1+\delta)^{-\alpha}\} + o(1)\{n^{-\alpha}L_1(n)\} \qquad \text{as} \quad n \to \infty.$$

On the other hand, we have $a_\nu \leqslant (\nu/n)^\eta\, a_n \leqslant (1+\delta)^\eta\, a_n$ for all $n \leqslant \nu \leqslant m$, then

$$\sum_{k=n}^{m} a_k \leqslant n\delta \left(1 + \frac{1}{n\delta}\right)(1+\delta)^\eta\, a_n\,,$$

and

$$n\delta\left(1+\frac{1}{n\delta}\right)(1+\delta)^{\eta}\, a_n \geqslant n^{-\alpha}L_1(n)\,\{1-(1+\delta)^{-\alpha}\} + o(1)\,\{n^{-\alpha}L_1(n)\}$$
$$\text{as} \quad n \to \infty.$$

It follows that for fixed δ, we have

$$\liminf_{n\to\infty} \frac{a_n}{n^{-\alpha-1}L_1(n)} \geqslant \frac{1-(1+\delta)^{-\alpha}}{(1+\delta)^{\eta}}.$$

Hence $\lim_{n\to\infty} \inf a_n \geqslant n^{-\alpha-1}L_1(n)$ by choosing δ arbitrarily small.

Therefore, as $n \to \infty$,

$$\begin{aligned} a_n &\simeq \alpha n^{-\alpha-1}L_1(n) \\ &\simeq \alpha n^{-\alpha-1}L(n), \end{aligned}$$

since $\lim_{n\to\infty} \sup a_n \leqslant \alpha n^{-\alpha-1}L_1(n)$ can be obtained by a similar argument.

LEMMA 2. *Suppose that* $a_n \geqslant 0$ *and* $\alpha \geqslant 0$. *Then, as* $n \to \infty$,

$$\sum_{k=n}^{\infty} a_k - \tfrac{1}{2}\, a_n \simeq n^{-\alpha}L(n)$$

if and only if

$$\sum_{k=n}^{\infty} a_k \simeq n^{-\alpha}L(n).$$

To prove Lemma 2, we only need to show that $a_n = o\{n^{-\alpha}L(n)\}$ in both cases. Actually,

$$\begin{aligned} a_n + a_{n+1} &= 2\left\{\sum_{k=n}^{\infty} a_k - \tfrac{1}{2}\, a_n - \sum_{k=n+1}^{\infty} a_k + \tfrac{1}{2}\, a_{n+1}\right\} \\ &= 2\left\{1-\left(1+\frac{1}{n}\right)^{-\alpha}\frac{L(n+1)}{L(n)} + o(1)\right\} n^{-\alpha}L(n) \\ &= o\{n^{-\alpha}L(n)\} \qquad \text{as} \qquad n \to \infty, \end{aligned}$$

or

$$\begin{aligned} a_n &= \sum_{k=n}^{\infty} a_k - \sum_{k=n+1}^{\infty} a_k \\ &= \left\{1-\left(1+\frac{1}{n}\right)^{-\alpha}\frac{L(n+1)}{L(n)} + o(1)\right\} n^{-\alpha}L(n) \\ &= o\{n^{-\alpha}L(n)\} \qquad \text{as} \qquad n \to \infty. \end{aligned}$$

Proof of the sufficiency of Theorems 1 and 2. We prove Theorem 2 for $1 < \alpha < 3$ first, and assume the asymptotic relation (3) for this range. Putting $A_n = \sum_{k=n}^{\infty} a_k$ and an adequate transformation on

$$\sum_{k=1}^{\infty} A_k\{\cos(k+1)x - \cos(k-1)x\}$$

[8, Lemma 2.1], we obtain the equality

$$-2\sum_{k=1}^{\infty} A_k \sin kx = \frac{1+\cos x}{\sin x}\{f(x) - f(0)\} - \sum_{k=1}^{\infty} a_k \sin kx, \tag{5}$$

and hence

$$-x\sum_{k=1}^{\infty}(A_k - \tfrac{1}{2}a_k)\sin kx \simeq f(x) - f(0) \qquad \text{as} \quad x \to +0.$$

This leads to

$$\sum_{k=1}^{\infty}\left(A_k - \frac{1}{2}a_k\right)\sin kx \simeq \frac{1}{\alpha-1}\,\frac{\pi}{2\Gamma(\alpha-1)\sin\frac{1}{2}(\alpha-1)\pi}\,x^{\alpha-2}L\left(\frac{1}{x}\right)$$

as $x \to +0$ by the assumption. Observing that $\{A_k - \frac{1}{2}a_k\}$ is monotone, by a known result [2, Theorem 1] we obtain

$$A_n - \frac{1}{2}a_n \simeq \frac{1}{\alpha-1}\,n^{1-\alpha}L(n) \qquad \text{as} \quad n \to \infty.$$

Finally, (1a) and (1b) follow from Lemmas 1 and 2.

We shall show that, for general α, Theorem 2 can be proved when Theorem 1 is known for $(\alpha - 1)$ and, alternately, Theorem 1 can be proved when Theorem 2 is known for $(\alpha - 1)$. The sufficiency of Theorems 1 and 2 will then follow since we have just shown, as a starting point, that the sufficiency of Theorem 2 is true for $1 < \alpha < 3$.

Now we assume the asymptotic relation (2) for $2i + 2 > \alpha > 2i$, $i \geqslant 1$, and that the sufficiency of Theorem 2 is known for $2i + 1 > \alpha' > 2i - 1$. From the equality

$$\sum_{k=1}^{\infty}\left\{A_k - \frac{1}{2}a_k\right\}\cos kx - \sum_{k=1}^{\infty}\left\{A_k - \frac{1}{2}a_k\right\} = \frac{1+\cos x}{2\sin x}\,g(x) - \sum_{k=1}^{\infty} ka_k,$$

which is the result from a transformation of

$$\sum_{k=1}^{\infty} A_k\{\sin(k+1)x - \sin(k-1)x\},$$

we write

$$\sum_{k=1}^{\infty} \left\{A_k - \frac{1}{2} a_k\right\} \cos kx - \sum_{k=1}^{\infty} \left\{A_k - \frac{1}{2} a_k\right\}$$

$$- \sum_{j=1}^{[\frac{\alpha}{2}-1]} \frac{(-1)^j}{(2j)!} \left\{ \sum_{k=1}^{\infty} k^{2j} \left(A_k - \frac{1}{2} a_k\right)\right\} x^{2j} \tag{6}$$

$$= \frac{1+\cos x}{2 \sin x} g(x) - \left[\sum_{k=1}^{\infty} k a_k + \sum_{j=1}^{[\frac{\alpha}{2}-1]} \frac{(-1)^j}{(2j)!} \left\{\sum_{k=1}^{\infty} k^{2j} \left(A_k - \frac{1}{2} a_k\right)\right\} x^{2j}\right]$$

$$= H_1(x) - H_2(x).$$

From the assumption that $g^{(2[\alpha/2]-1)}(0)$ exists, $\sum_{k=1}^{\infty} k^{2[\alpha/2]-1} a_k \sin kx$ converges uniformly and absolutely on $[0, \pi]$. Also $g(x)$ can be expanded according to Taylor's formula such that

$$g(x) = \sum_{h=1}^{[\frac{\alpha}{2}]-1} \frac{g^{(2h-1)}(0)}{(2h-1)!} x^{2h-1} + \frac{g^{\left(2[\frac{\alpha}{2}]-1\right)}(x_1)}{\left(2\left[\frac{\alpha}{2}\right]-1\right)!} (x - x_1)^{2[\frac{\alpha}{2}]-1}, \quad 0 < x_1 < x,$$

$$= \sum_{h=1}^{[\frac{\alpha}{2}]-1} \frac{(-1)^{h-1}}{(2h-1)!} \left(\sum_{k=1}^{\infty} k^{2h-1} a_k\right) x^{2h-1} + C(g, \alpha)(x - x_1)^{2[\frac{\alpha}{2}]-1},$$

where $C(g, \alpha)$ is a constant depending on g and α. Hence, as $x \to +0$,

$$H_1(x) = \left\{\frac{1}{x} - \sum_{n=1}^{\infty} \frac{B_n}{(2n)!} x^{2n-1}\right\} g(x) \text{ (B_n's are Bernoulli's constants [9])}$$

$$= \frac{1}{x} g(x) + \left\{\sum_{n=1}^{[\frac{\alpha}{2}]-1} \frac{B_n}{(2n)!} x^{2n-1}\right\} \left\{\sum_{h=1}^{[\frac{\alpha}{2}]-1} \frac{(-1)^h}{(2h-1)!} \left(\sum_{k=1}^{\infty} k^{2h-1} a_k\right) x^{2h-1}\right\}$$

$$+ o\left\{x^{\alpha-2} L\left(\frac{1}{x}\right)\right\}$$

$$= \frac{1}{x} g(x) + \sum_{j=1}^{[\frac{\alpha}{2}]-1} \left\{\sum_{r=1}^{j} \frac{B_r}{(2r)!} \frac{(-1)^{j-r+1}}{(2j-2r+1)!} \left(\sum_{k=1}^{\infty} k^{2j-2r+1} a_k\right) x^{2j}\right\}$$

$$+ o\left\{x^{\alpha-2} L\left(\frac{1}{x}\right)\right\}.$$

On the other hand,

$$H_2(x) = \sum_{k=1}^{\infty} ka_k + \sum_{j=1}^{[\frac{\alpha}{2}]-1} \frac{(-1)^j}{(2j)!} \left\{ \sum_{k=1}^{\infty} \left[\sum_{y=1}^{k} y^{2j} - \frac{1}{2} k^{2j} \right] a_k \right\} x^{2j}$$

$$= \frac{1}{x} \sum_{j=1}^{j=[\frac{\alpha}{2}]} \frac{1}{(2j-1)!} x^{2j-1} g^{(2j-1)}(0)$$

$$+ \sum_{j=1}^{[\frac{\alpha}{2}]-1} \frac{1}{(2j)!} \left\{ \sum_{r=1}^{j} (-1)^{j+r+1} \frac{B_r}{(2r)} \binom{2j}{2r-1} \left(\sum_{k=1}^{\infty} k^{2j-2r+1} a_k \right) x^{2j} \right\},$$

since [9, p. 8]

$$\sum_{y=1}^{k} y^{2j} = \frac{k^{2j+1}}{2j+1} + \frac{k^{2j}}{2} + \sum_{r=1}^{j} (-1)^{r-1} \frac{B_r}{2r} \binom{2j}{2r-1} k^{2j-2r+1}$$

and

$$g^{(2j-1)}(0) = (-1)^{j-1} \sum_{k=1}^{\infty} k^{2j-1} a_k .$$

Therefore, the right hand side of (6) gives:

$$H_1(x) - H_2(x) = \frac{1}{x} \left\{ g(x) - \sum_{j=1}^{[\frac{\alpha}{2}]} \frac{x^{2j-1}}{(2j-1)!} g^{(2j-1)}(0) \right\} + o \left\{ x^{\alpha-2} L\left(\frac{1}{x}\right) \right\}$$

$$\simeq \frac{\pi}{2\Gamma(\alpha)\sin(\alpha\pi/2)} x^{\alpha-2} L\left(\frac{1}{x}\right) \qquad \text{as} \quad x \to +0.$$

This leads to

$$\sum_{k=1}^{\infty} \left(A_k - \frac{1}{2} a_k \right) \cos kx - \sum_{k=1}^{\infty} \left(A_k - \frac{1}{2} a_k \right)$$

$$- \sum_{j=1}^{[\frac{\alpha}{2}]-1} \frac{(-1)^j}{(2j)!} \left\{ \sum_{k=1}^{\infty} k^{2j} \left(A_k - \frac{1}{2} a_k \right) \right\} x^{2j}$$

$$\simeq \frac{1}{\alpha-1} \frac{\pi}{2\Gamma(\alpha)\sin(\alpha\pi/2)} x^{\alpha-2} L\left(\frac{1}{x}\right) \qquad \text{as} \quad x \to +0.$$

Finally, the sufficiency of Theorem 1 follows from the assumption that the sufficiency of Theorem 2 is known for $2i+1 > \alpha' = \alpha - 1 > 2i - 1$.

To prove Theorem 2 (the sufficiency) for general α, $2i+3>\alpha>2i+1$ and $i>1$, we assume the asymptotic relation (3) for this range and also that Theorem 1 (the sufficiency) is known for $2i+2>\alpha'>2i$. From (5) we may write

$$\begin{aligned}\sum_{k=1}^{\infty}\left(A_k-\frac{1}{2}a_k\right)\sin kx &- \sum_{j=1}^{[\frac{\alpha}{2}-\frac{1}{2}]}\frac{1}{(2j-1)!}(-1)^{j-1}\\ &\times\left\{\sum_{k=1}^{\infty}k^{2j-1}\left(A_k-\frac{1}{2}a_k\right)\right\}x^{2j-1}\\ = -\frac{1+\cos x}{2\sin x}\{f(x)-f(0)\}&\\ +\sum_{j=1}^{[\frac{\alpha-1}{2}]}\frac{(-1)^j}{(2j-1)!}&\left\{\sum_{k=1}^{\infty}k^{2j-1}\left(A_k-\frac{1}{2}a_k\right)\right\}x^{2j-1}\\ = -H_3(x)+H_4(x).&\end{aligned} \tag{7}$$

From the assumption that $f^{(2[(\alpha-1)/2])}(0)$ exists,

$$\sum_{k=1}^{\infty}k^{2[\frac{\alpha-1}{2}]-1}A_k<\infty$$

and $f(x)$ may be expanded according to Taylor's formula such that

$$\begin{aligned}f(x)=f(0)+&\sum_{j=1}^{[\frac{\alpha-1}{2}]-1}\frac{f^{(2j)}(0)}{(2j)!}x^{2j}\\ &+\frac{f^{(2[\frac{\alpha-1}{2}])}(x_1)}{\left(2\left[\frac{\alpha-1}{2}\right]\right)!}(x-x_1)^{2[\frac{\alpha-1}{2}]},\quad 0<x_1<x.\end{aligned}$$

Hence, as $x\to+0$, we have

$$\begin{aligned}H_3(x)=\frac{1}{x}\{f(x)-f(0)\}&-\frac{1}{x}\left\{\sum_{n=1}^{[\frac{\alpha-3}{2}]}\frac{B_n}{(2n)!}x^{2n}\right\}\\ \times&\left\{\sum_{h=1}^{[\frac{\alpha-3}{2}]}\frac{(-1)^h\sum_{k=1}^{\infty}k^{2h}a_k}{(2h)!}x^{2h}\right\}+o\left\{x^{\alpha-2}L\left(\frac{1}{x}\right)\right\}\end{aligned}$$

$$= \frac{1}{x}\{f(x)-f(0)\} - \frac{1}{x}\left\{\sum_{m=0}^{[\frac{\alpha-5}{2}]} \frac{B_{m+1}}{(2m+2)!} x^{2m+2}\right\}$$

$$\times \left\{\sum_{p=0}^{[\frac{\alpha-5}{2}]} \frac{(-1)^{p+1}\sum_{k=1}^{\infty} k^{2p+2}a_k}{(2p+2)!} x^{2p+2}\right\} + o\left\{x^{\alpha-2}L\left(\frac{1}{x}\right)\right\}$$

$$= \frac{1}{x}\{f(x)-f(0)\}$$

$$-\frac{1}{x}\sum_{s=0}^{[\frac{\alpha-5}{2}]}\left\{\sum_{m=0}^{s} \frac{B_{m+1}}{(2m+2)!} \frac{(-1)^{s-m+1}\sum_{k=1}^{\infty} k^{2s-2m+2}a_k}{(2s-2m+2)!}\right\} x^{2s+4}$$

$$+ o\left\{x^{\alpha-2}L\left(\frac{1}{x}\right)\right\}$$

$$= \frac{1}{x}\{f(x)-f(0)\}$$

$$-\frac{1}{x}\sum_{j=2}^{[\frac{\alpha-1}{2}]}\left\{\sum_{r=1}^{j-1}\frac{B_r}{(2r)!}(-1)^{j-r}\frac{\sum_{k=1}^{\infty} k^{2j-2r}a_k}{(2j-2r)!}\right\} x^{2j} + o\left\{x^{\alpha-2}L\left(\frac{1}{x}\right)\right\};$$

and, by

$$\sum_{k=1}^{\infty} k^{2j-1}A_k = \sum_{k=1}^{\infty}\left(\sum_{y=1}^{k} y^{2j-1}\right) a_k$$

$$= \sum_{k=1}^{\infty}\left(\frac{k^{2j}}{2j} + \frac{1}{2}k^{2j-1} + \sum_{r=1}^{j-1}(-1)^{r-1}\frac{B_r}{2r}\binom{2j-1}{2r-1}k^{2j-2r}\right) a_k \qquad \text{for} \quad j \geqslant 2$$

and

$$\sum_{k=1}^{\infty} kA_k = \sum_{k=1}^{\infty}\left(\frac{k^2}{2} + \frac{1}{2}k\right) a_k,$$

we have

$$H_4(x) = \frac{1}{x}\sum_{j=1}^{[\frac{\alpha-1}{2}]}(-1)^j\frac{1}{(2j)!}\left(\sum_{k=1}^{\infty} k^{2j}a_k\right) x^{2j}$$

$$\frac{1}{x}\sum_{j=2}^{[\frac{\alpha-1}{2}]}\frac{(-1)^j}{(2j-1)!}\left\{\sum_{r=1}^{j-1}(-1)^{r-1}\frac{B_r}{2r}\binom{2j-1}{2r-1}\sum_{k=1}^{\infty} k^{2j-2r}a_k\right\} x^{2j}.$$

Then, as $x \to +0$, the right hand side of (7) gives:

$$-H_3(x) + H_4(x) \simeq -\frac{\pi}{2\Gamma(\alpha)\cos(\alpha\pi/2)} x^{\alpha-2} L\left(\frac{1}{x}\right)$$
$$\simeq \frac{\pi}{2(\alpha-1)\,\Gamma(\alpha-1)\sin\frac{\alpha-1}{2}\pi} x^{\alpha-2} L\left(\frac{1}{x}\right).$$

Now, from (7) and the assumption that the sufficiency of Theorem 1 is known for $2i + 2 > \alpha' > 2i$, we obtain

$$A_n - \frac{1}{2} a_n \simeq \frac{1}{\alpha - 1} n^{1-\alpha} L(n) \qquad \text{as} \quad n \to \infty,$$

and hence (1a).

Proof of the necessity of Theorems 1 and 2. We may rewrite Eqs. (6) and (7) in the following form (subject to the existence of $g^{(2[\alpha/2]-1)}(0)$ and $f^{(2[(\alpha-1)/2])}(0)$):

$$\sum_{k=1}^{\infty} \left\{A_k - \frac{1}{2} a_k\right\} \cos kx - \sum_{j=0}^{[\frac{\alpha}{2}-1]} \frac{1}{(2j)!} F^{(2j)}(0)\, x^{2j}$$
$$= \frac{1}{x} \left\{g(x) - \sum_{j=1}^{[\frac{\alpha}{2}]} \frac{1}{(2j-1)!} g^{(2j-1)}(0)\, x^{2j-1}\right\} + o\left\{x^{\alpha-2} L\left(\frac{1}{x}\right)\right\} \quad \text{as} \quad x \to +0 \tag{8}$$

and

$$\sum_{k=1}^{\infty} \left\{A_k - \frac{1}{2} a_k\right\} \sin kx - \sum_{j=1}^{[\frac{\alpha-1}{2}]} \frac{1}{(2j-1)!} G^{(2j-1)}(0)\, x^{2j-1}$$
$$= -\frac{1}{x} \left\{f(x) - \sum_{j=0}^{[\frac{\alpha-1}{2}]} \frac{1}{(2j)!} f^{(2j)}(0)\, x^{2j}\right\} + o\left\{x^{\alpha-2} L\left(\frac{1}{x}\right)\right\} \quad \text{as} \quad x \to +0, \tag{9}$$

where

$$G^{(2j-1)}(0) = (-1)^{j-1} \sum_{k=1}^{\infty} k^{2j-1}(A_k - \tfrac{1}{2} a_k)$$

and

$$F^{(2j)}(0) = (-1)^{j} \sum_{k=1}^{\infty} k^{2j}(A_k - \tfrac{1}{2} a_k).$$

Then our proof of the necessity of the theorems can be carried on in a similar process as in the proof of the sufficiency. When we prove Theorem 1 for general α, we assume (1a) for $2i + 2 > \alpha > 2i$ and the truth of the necessity of Theorem 2 for $2i + 1 > \alpha' = \alpha - 1 > 2i - 1$. Since $\sum_{k=n}^{\infty} a_k = 0(n^{1-\alpha+\epsilon})$ implies that $\sum_{k=1}^{\infty} k^{2[\alpha/2]-1} a_k < \infty$ for $\alpha \neq 2i$, hence $g^{(2[\alpha/2]-1)}(0)$ exists. Then Eq. (8) is applicable. Observing that

$$A_n \simeq \frac{1}{\alpha'} n^{-\alpha'} L(n) \qquad \text{as} \quad n \to \infty$$

is equivalent to

$$\sum_{k=n}^{\infty} \left(A_k - \frac{1}{2} a_k\right) \simeq \frac{1}{\alpha'(\alpha' - 1)} n^{1-\alpha'} L(n) \qquad \text{as} \quad n \to \infty,$$

and

$$\sum_{j=0}^{[\frac{\alpha}{2}-1]} \frac{1}{(2j)!} F^{(2j)}(0)\, x^{2j} = \sum_{j=0}^{[\frac{\alpha'-1}{2}]} \frac{1}{(2j)!} F^{(2j)}(0)\, x^{2j},$$

we find that the left hand side of (8), and hence the right hand side of (8), is asymptotically equal to

$$\frac{\pi}{(\alpha - 1)\, 2\Gamma(\alpha - 1) \cos \dfrac{\alpha - 1}{2} \pi} x^{\alpha-2} L\left(\frac{1}{x}\right) \qquad \text{as} \quad x \to +0.$$

This leads to what we intended to prove. To prove the necessity of Theorem 2 for general α, we may assume (1a) for $2i + 3 > \alpha > 2i + 1$ and the truth of the necessity of Theorem 1 for $2i + 2 > \alpha' > 2i$. The remainder of the process is similar. It is already known in the literature [2] that, for $0 < \alpha - 1 < 2$,

$$\sum_{k=1}^{\infty} \left(A_k - \frac{1}{2} a_k\right) \sin kx \simeq \frac{\pi}{(\alpha - 1)\, 2\Gamma(\alpha - 1) \sin \dfrac{\alpha - 1}{2} \pi} x^{\alpha-2} L\left(\frac{1}{x}\right) \qquad \text{as} \quad x \to +0,$$

if

$$A_n \simeq \frac{1}{\alpha - 1} n^{1-\alpha} L(n) \qquad \text{as} \quad n \to \infty,$$

since $\{A_n - \frac{1}{2} a_n\}$ is monotone. Thus we start from the truth of the necessity of Theorem 2 for $1 < \alpha < 3$, which can be obtained by the said known result and Eq. (9). The proof is completed.

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